

Easy Complex Number Manipulation With MBASIC

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This article shows how to use MBASIC to manipulate complex scalars and matrices.

I. Introduction

The interpretive MBASIC language processor does not have complex data types in its present implementation at JPL on the Univac 1108 systems. Nevertheless, complex arithmetic can easily be simulated by using the MBASIC matrix handling ability. The method is fast because it uses the machine-coded matrix arithmetic subroutines of the processor; it is convenient because subscript book-keeping is avoided. After a small amount of machinery has been set up, complex expressions can be written almost as if actual complex data types were being used. A disadvantage of the method is that 4 words of storage are needed for each complex number, rather than 2.

II. Scalars

The complex number system can be simulated by a class of 2 by 2 matrices. The identification is:

$$a + bi \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \text{ } a \text{ and } b \text{ real.}$$

Thus

$$1 \leftrightarrow \text{REU} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (real unit),}$$

$$i \leftrightarrow \text{IMU} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ (imaginary unit),}$$

$$0 \leftrightarrow \text{ZERO} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $z \leftrightarrow Z$ and $w \leftrightarrow W$, then $\bar{z} \leftrightarrow \text{TRN}(Z)$ and $z + w \leftrightarrow Z + W$. Also, $zw \leftrightarrow Z^*W$, for if $z = a + ib$, $w = c + id$, then

$$zw = ac - bd + i(ad + bc),$$

$$\begin{aligned} Z^*W &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} \end{aligned}$$

If $z \neq 0$, then $1/z \leftrightarrow \text{INV}(Z)$, $w/z \leftrightarrow W * \text{INV}(Z)$.

The squared magnitude $|z|^2$ can be obtained as a scalar real by calling $\text{DET}(Z)$, or as a complex real by using:

$$Z * \text{TRN}(Z) = \begin{bmatrix} |z|^2 & 0 \\ 0 & |z|^2 \end{bmatrix}.$$

Example: A digital filter has the z -transform transfer function

$$h(z) = 1 + \left(\frac{1-z}{2z-1} \right)^3.$$

The following MBASIC program prints $|h(e^{i2\pi f})|^2$ vs f for $f = 0, 0.05, 0.1, \dots, 0.5$ Hz:

```
>LIST
10 REAL REU(2,2)/IDN(2)/, IMU(2,2)/0,-1,1,0/,
    Z(2,2) H(2,2)

20 PRINT 'HZ','RESPONSE' \ HZ, DET(H) WHERE
    HZ=K/20, OMEGA=2*PI*HZ, Z=COSR(OMEGA)
    *REU + SINR(OMEGA)*IMU, H=REU + ((REU
    -Z)*INV(2*Z-REU))**3 FOR K=0 TO 10

>RUN
HZ          RESPONSE
0           1
.5E-01      1.0460479
.1          1.1580165
.15         1.087936
.2          .92637333
.25         .776
.3          .66241923
.35         .58356814
.4          .53269148
.45         .50431055
.5          .49519892
>
```

III. Matrices

A powerful extension of the technique allows arithmetic with complex matrices without resorting to subscript manipulation. With the complex m -by- n matrix $A+iB$, associate the real partitioned $2m$ by $2n$ matrix

$$E = \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$$

which can be constructed by a declaration:

```
>REAL M=2, N=3, A(M,N)/1,2,3,4,5,6/, B(M,N)/7,
    & 8,9,10,11,12/
>REAL E1(2*N,M)/TRN(A),TRN(-B)/,&
    E2(2*N,M)/TRN(B),TRN(A)/,&
    E(2*M,2*N)/TRN(E1),TRN(E2)/
>PRINT USING '(6(4%)/)': E
  1  2  3 -7 -8 -9
  4  5  6 -10 -11 -12
  7  8  9  1  2  3
 10 11 12  4  5  6
>
```

Similarly, A and B can be reconstructed from E :

```
>REAL ZERO(N,N), REPT(2*N,N)/IDN(N)/,&
    IMPT(2*N,N)/ZERO,-IDN(N)/
>REAL E3(M,2*N)/E/, A(M,N)/E3*REPT/,&
    B(M,N)/E3*IMPT/
>PRINT A:\\B:
  1  2  3  4  5  6
  7  8  9 10 11 12
>
```

Complex matrices can be added and multiplied (if conformable) by doing the operations on their E -matrices. In particular, a nonsingular n -by- n complex matrix can be inverted by using the function INV on its E -matrix.

We can also associate a complex n -vector $X+iY$ with the column vector

$$\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then the multiplication $(A+iB)(X+iY)$ can be executed by performing

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX - BY \\ BX + AY \end{bmatrix}$$

Example. The following MBASIC program solves complex n -by- n linear systems $AX=B$. It is run with the example

$$A = \begin{bmatrix} 1 & i \\ 2-i & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1+i \\ 0 \end{bmatrix}$$

```

>LIST
10 INPUT USING 'DIMENSION: #':N
20 REAL AR(N,N), AI(N,N), BR(N), BI(N)
30 PRINT 'INPUT REAL PART OF A, THEN IMAG-
  INARY PART'
40 INPUT AR,AI
50 PRINT 'INPUT REAL PART OF B, THEN IMAG-
  INARY PART'
60 INPUT BR,BI
70 REAL A1(2*N,N)/TRN(AR),TRN(-AI)/,
  A2(2*N,N)/TRN(AI),TRN(AR)/,
  A(2*N,2*N)/TRN(A1),TRN(A2)/,
  B(2*N)/BR,BI/, X(2*N)/INV(A)*B/
80 PRINT 'REAL PART OF SOLUTION' \X(K); FOR
  K=1 TO N \ 'IMAGINARY PART OF SOLU-
  TION' \X(K); FOR K=N+1 TO 2*N

>RUN
DIMENSION: 2
INPUT REAL PART OF A, THEN IMAGINARY PART
?1,0,2,1
?0,1,-1,0
INPUT REAL PART OF B, THEN IMAGINARY PART
?1,0
?1,0
REAL PART OF SOLUTION
-.49999998 .5
IMAGINARY PART OF SOLUTION
.49999999 -1.5
>

```

It is also possible, though perhaps not as convenient, to associate the m -by- n complex matrix $(a_{jk} + ib_{jk})$ with the real $2m$ -by- $2n$ matrix

$$F = \begin{vmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & & \\ Z_{m1} & \cdots & Z_{mn} \end{vmatrix}$$

where Z_{jk} is the 2-by-2 matrix we were associating with the complex number $a_{jk} + ib_{jk}$. These F -matrices can be manipulated just like the E -matrices. One can go from one to the other by row and column permutations. In this connection, we mention that

$$\text{DET}(E) = \text{DET}(F) = |\det(A + iB)|^2.$$

Is there a way to compute $\det(A + iB)$ in the same spirit?

R. J. Hanson showed that if $A + iB$ is Hermitian, then its E matrix has the same eigenvalues, with the multiplicities doubled (Ref. 1). L. W. Ehrlich (Ref. 2) showed that inversion of $A + iB$ with complex arithmetic can be carried out faster and more accurately than inversion of E with real arithmetic. Of course, if the complex inversion has to be coded step by step in the source language of an interpretive processor, it cannot compete in speed with real inversion of E by a machine-language subroutine.

References

1. Hanson, R. J., *Remarks on the Calculation of Eigenvalues and Eigenvectors for Complex Hermitian Matrices*, Technical Memorandum 205, Jet Propulsion Laboratory, Pasadena, Calif., October, 1968.
2. Ehrlich, L. W., "Complex matrix inversion versus real," *Communications of the ACM*, Vol. 13, pp. 561-62, 1970.